Spatio-temporal Holling type-IV and Leslie type model: existence and non-existence of spatial pattern

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Outline

Outline

Temporal model

$$\begin{aligned} \frac{dX}{dT} &= rX\left(1-\frac{X}{k}\right)-\frac{mXY}{b+X^2} \\ \frac{dY}{dT} &= Y\left[s\left(1-\frac{Y}{hX}\right)\right] \\ X(0) &> 0,\ X(0) > 0. \end{aligned}$$

- r: Intrinsic growth rate of prey in the absence of predation
- k: Environmental carrying capacity
- m: per capita rate of predation of the predator
- b: half saturation constant
- h: A measure of the food quality of the prey for conversion into predator births
- s: Intrinsic growth rate of predator

Non-dimesionalised temporal model

$$\begin{aligned}\frac{dx}{dt} &= x(1-x) - \frac{xy}{a+x^2}\\ \frac{dy}{dt} &= y(\delta - \beta\frac{y}{x})\\ x(0) &> 0,\ y(0) > 0.\end{aligned}$$

where,

$$t = rT,\ x = \frac{X}{k},\ y = \frac{mY}{rk},\ a = \frac{b}{k^2},\ \delta = \frac{s}{r},\ \beta = \frac{sk}{hm}$$

Equilibria, stability and Bifurcations

Equilibria

Let $\Delta_1 = 1 - 3(a + \frac{\delta}{\beta})$, $\Delta_2 = (1 - 27a - 3\Delta_1)^2 - 4\Delta_1^3$

- The model has one boundary equilibrium point $E_1(1, 0)$ which is a saddle point.
- If $\Delta_2 > 0$ or $\Delta_2 = 0$ & $\Delta_1 = 0 \Rightarrow$ unique interior equilibrium E^*.
- If $\Delta_2 = 0$ & $\Delta_1 > 0 \Rightarrow E^*$ and E_1^* or E^* and E_2^*.
- If $\Delta_2 < 0 \Rightarrow E_1^*, E_2^*, E_3^*$.

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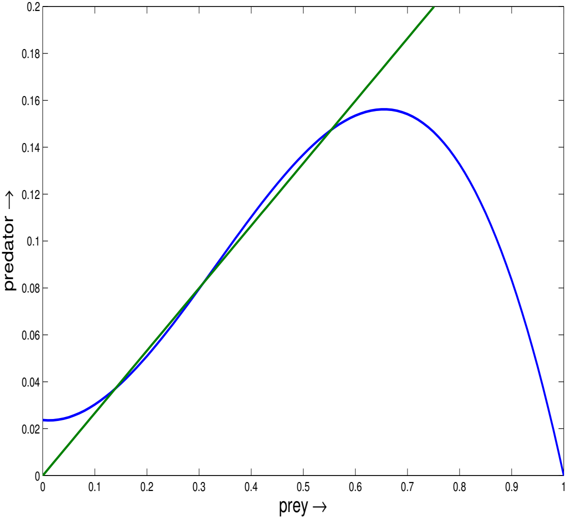
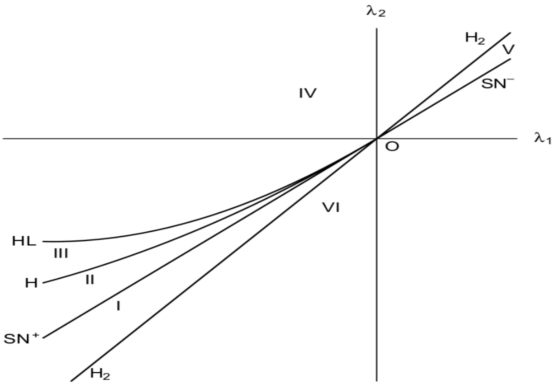


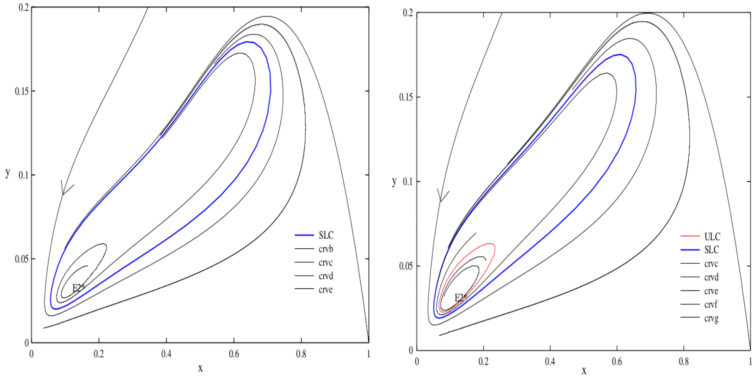
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
predator →
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
prey →

Results from Xiao. et.al.

For the parameter set $(a, \delta, \beta) = (0.0236653, 0.561553, 2.03078)$ the model exhibits Bogdanov-Takens bifurcation.



http://dx.doi.org/10.1016/j.chaos.2006.03.068



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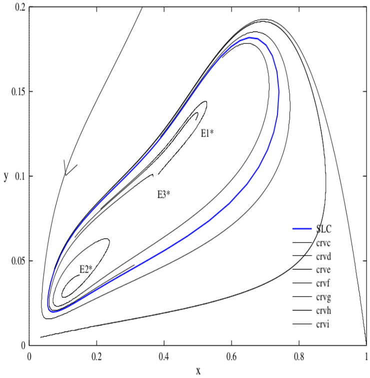

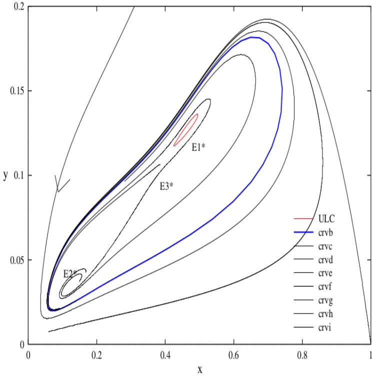

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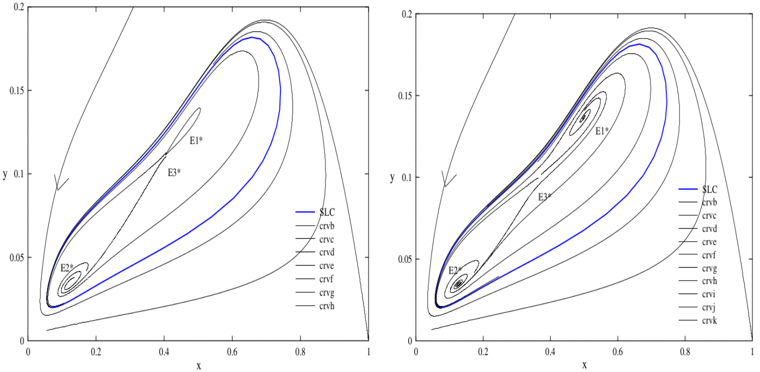

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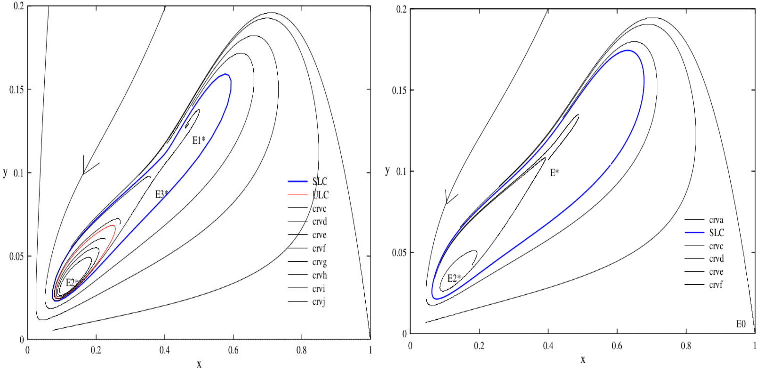

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Outline

spatial Model

$$\begin{aligned}\frac{\partial u}{\partial t} &= u(1-u) - \frac{uv}{a+u^2} + d_1 \Delta u, \quad x \in \Omega, t > 0\\ \frac{\partial v}{\partial t} &= v(\delta - \beta \frac{v}{u}) + d_2 \Delta v, \quad x \in \Omega, t > 0\end{aligned}$$

subjected to positive initial conditions:

$$u(x,0) \equiv u_0(x) > 0, v(x,0) \equiv v_0(x) > 0, \; x \in \Omega$$

and zero-flux boundary conditions:

$$\frac{\partial u}{\partial \nu} = 0, \; \frac{\partial v}{\partial \nu} = 0 \;\; x \in \partial\Omega$$

Unique homogeneous steady state ($\Delta_2 > 0$ or $\Delta_2 = 0 \& \Delta_1 = 0$)

The spatial system is persistence if $a\beta > \delta$

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The spatial system is persistence if $a\beta > \delta$

- $\limsup_{t\to\infty} \max_{\overline{\Omega}} u(\cdot, t) \leq 1$
- $\limsup_{t\to\infty} \max_{\overline{\Omega}} v(\cdot, t) \leq \frac{\delta}{\beta}$
- $\liminf_{t\to\infty} \min_{\overline{\Omega}} u(\cdot, t) \geq 1 - \frac{\delta}{a\beta}$
- $\liminf_{t\to\infty} \min_{\overline{\Omega}} v(\cdot, t) \geq \frac{\delta K}{\beta}$

Let $M = \max\{1, \frac{\delta}{\beta}\}$ and $m = \min\{K, \frac{\delta K}{\beta}\}$ and $K = 1 - \frac{\delta}{a\beta}$. Hence, the system is persistence when $K > 0$ which implies $a\beta > \delta$.

Global stability of the unique homogeneous steady state

Theorem: The interior equilibrium point is globally stable if

$(a+K^2)^2 > \frac{2\delta}{\beta}$ & $K > \frac{\frac{\delta^2}{4}+\frac{4\delta}{a^2}+\frac{1}{a^2}}{2\beta+\frac{\delta}{a+1}}$.

Proof. We have taken the Lyapunov function,

$$E(t) = \int_\Omega W(u,v)dx$$

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Proof. We have taken the Lyapunov function,

$$E(t) = \int_\Omega W(u,v)dx$$
$$W(u,v) = \int \frac{u^2-u_*^2}{u^2}du + \int \frac{v-v_*}{v}dv$$

$$\frac{E(t)}{dt} = E_1(t) + E_2(t)$$

$$E_1(t) = -\int_\Omega \left\{ d_1 \frac{2u_*}{u^3}|\nabla u|^2 + d_2 \frac{v_*}{v^2}|\nabla v|^2 \right\} dx \leq 0$$

Global stability of the unique homogeneous steady state

$$E_2(t) = -\int_{\Omega} \begin{pmatrix} \xi & \eta \end{pmatrix} \begin{pmatrix} p & q \\ q & r \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} dx$$

where,

$$\begin{aligned} p &= \frac{u+u_*}{u}\left[1-\frac{v_*(u+u_*)}{(a+u_*^2)(a+u^2)}\right] \\ q &= \frac{1}{2u}\left[\frac{u+u_*}{a+u^2}-\delta\right] \\ r &= \frac{\beta}{u}. \end{aligned}$$

The spatial system is globally Stable if $\begin{pmatrix} p & q \\ q & r \end{pmatrix}$ is positive definite.

Hence the theorem.

Existence and Non-existence of non-constant steady states

$$
\begin{aligned}
-d_1 \Delta u &= u(1-u) - \frac{uv}{a+u^2}, \quad x \in \Omega, && (1)\\
-d_2 \Delta v &= v(\delta - \beta \frac{v}{u}), \quad x \in \Omega, && (2)\\
\frac{\partial u}{\partial \nu} &= 0, \ \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega. && (3)
\end{aligned}
$$

Let $0 = \mu_0 < \mu_1, \mu_2, \cdots \to \infty$ be the eigenvalues of the laplacian operator Δ on Ω under the given boundary condition.

Non-existence of non-constant steady states

Theorem: If $d_2 > \frac{\delta}{\mu_1}$ then there exists a constant d_1^* depending on d_2 such that the system has no positive non-constant solution when $d_1 \geq d_1^*$.

Proof. Assume that (u, v) is a positive solution of the model. Let $\overline{u} = \frac{1}{|u|}\int_\Omega u dx$ for any $u \in L^1(\Omega)$.

$$\int_\Omega d_1|\nabla u|^2 dx \leq \int_\Omega \left[\alpha(u-\overline{u})^2 + |u-\overline{u}||v-\overline{v}|\right] dx$$

$$\int_\Omega d_2|\nabla v|^2 dx \leq \int_\Omega \left[\frac{\beta\overline{v}^2}{u\overline{u}}(u-\overline{u})(v-\overline{v}) + \delta(v-\overline{v})^2\right] dx$$

$$\int_\Omega [d_1|\nabla u|^2 + d_2|\nabla v|^2] dx \leq \left[\left(\alpha + \frac{\nu}{\epsilon}\right)(u-\overline{u})^2 + (\delta + \nu\epsilon)(v-\overline{v})^2\right] dx.$$

for some positive constant ν and an arbitrary small number ϵ.

Non-existence of non-constant steady states

Now by using Poinćare inequality,

$$\int_\Omega \mu_1[d_1|u-\overline{u}|^2 + d_2|v-\overline{v}|^2]dx$$

$$\leq \int_\Omega \left[\left(\alpha + \frac{\nu}{\epsilon}\right)(u-\overline{u})^2 + (\delta + \nu\epsilon)(v-\overline{v})^2\right] dx.$$

As $d_2\mu_1 > \delta$ there exists $\epsilon_0 > 0$ such that $d_2\mu_1 > \delta + \nu\epsilon_0$ and we take $d_1^* = \frac{1}{\mu_1}\left(\alpha + \frac{\nu}{\epsilon}\right)$. Therefore we get $u = \overline{u}$ and $v = \overline{v}$.

Turing Instability

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\begin{aligned}
a_{11} + a_{22} &< 0, \; a_{11}a_{22} - a_{12}a_{21} > 0 \\
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which leads to the condition

$$
\frac{d_2}{d_1} > \frac{a_{11}a_{22} - 2a_{12}a_{21} + 2\sqrt{-a_{12}a_{21}(a_{11}a_{22} - a_{12}a_{21})}}{a_{11}^2}
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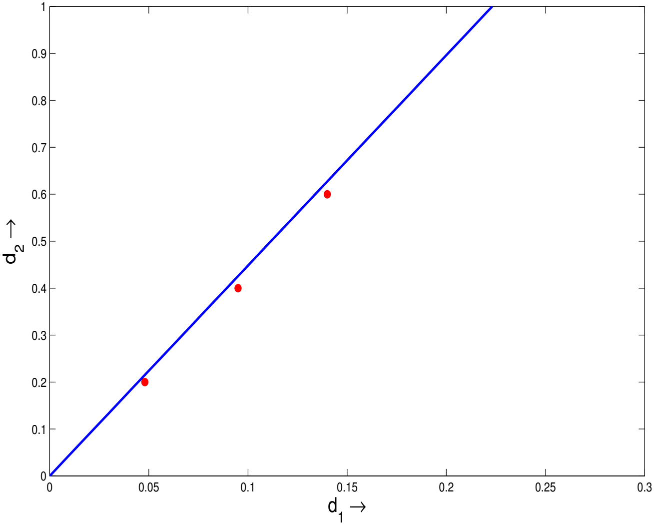
Around the wave number $k_{min}^2 = \frac{(a_{11}d_2 + a_{22}d_1)}{2d_1d_2}$. Here

$$\begin{aligned} a_{11} &= u\left(\frac{2\delta u^2}{\beta(a+u^2)^2} - 1\right)\Big|_{E^*}, \; a_{12} = -\frac{u}{a+u^2}\Big|_{E^*}, \\ a_{21} &= \frac{\delta^2}{\beta}, \; a_{22} = -\delta. \end{aligned}$$

Turing Instability



Existence of non-constant steady states

Let,

$$
\begin{aligned}
\mu_{\pm}(d_1, d_2) &= \frac{d_2 a_{11} - d_1\delta \pm \sqrt{(d_2 a_{11} - d_1\delta)^2 + 4d_1 d_2 \delta(a_{11} + \frac{\delta a_{12}}{\beta})}}{2d_1 d_2} \\
A(d_1, d_2) &= \{\mu : \mu \geq 0, \mu_-(d_1, d_2) < \mu < \mu_+(d_1, d_2)\} \\
S_p &= \{\mu_0, \mu_1, \mu_2, \cdots\} \\
\sigma &= \begin{pmatrix} \sum_{\mu_i \in A\cap S_p} m(\mu_i) & \text{if} A \cup S_p \neq \emptyset \\ 0 & \text{if} A \cup S_p = \emptyset \end{pmatrix}
\end{aligned}
$$

$m(\mu_i)$ the multiplicity of μ_i. Then by Leray-Schauder degree theory we prove the following result.

Existence of non-constant steady states

If the model has unique homogeneous steady state and if $\frac{a_{11}}{d_1} \in (\mu_k, \mu_{k+1})$ for some $k \geq 1$ with $\sigma_k = \sum_i^k m(\mu_i)$ being odd then there exists a positive constant d^* such that the system has at least one non-constant positive solution for all $d \geq d^*$.

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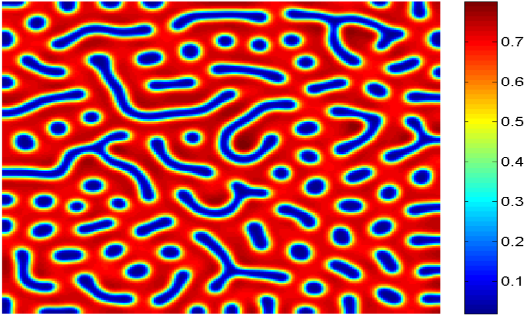

Spatial Hopf bifurcation

Theorem: The system undergoes hopf bifurcation at the same parametric threshold as for the temporal system in the case of unique interior equilibrium point having the same direction.

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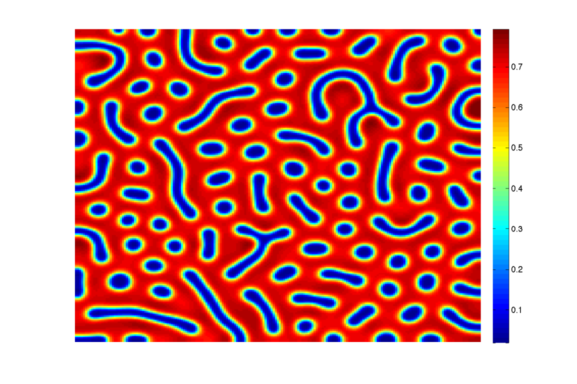

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- Existence of Travelling wave solution
- Introduction of Alee-effect to the prey growth

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References I

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Thank You