Harmless maturation delay in prey-predator type interactions

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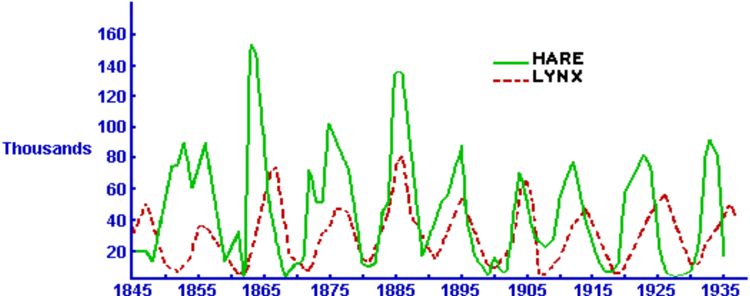
Outline

1. Basic prey-predator model
2. Delayed models
3. Model with Allee effect
4. Effect of maturation delay

Outline

1. Basic prey-predator model
2. Delayed models
3. Model with Allee effect
4. Effect of maturation delay

Lynx-Hare Data
Example of prey-predator interaction



Basic prey-predator models
Rosenzweig-MacArthur model

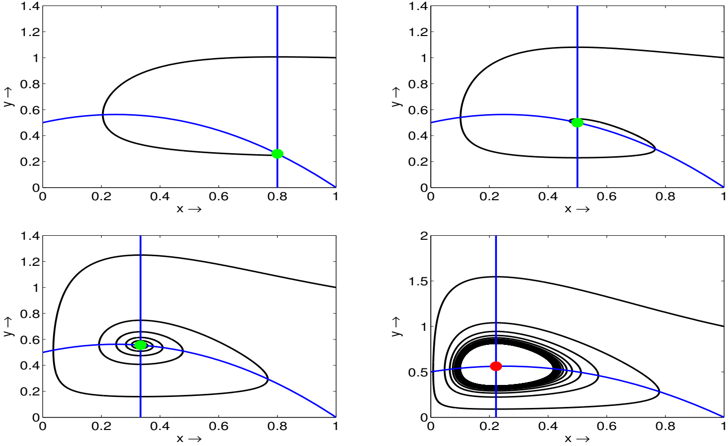
RM model

$$\begin{aligned} \frac{dx}{dt} &= x(1-x) - \frac{xy}{a+x} \equiv F(x,y) \\ \frac{dy}{dt} &= \frac{bxy}{a+x} - cy \equiv G(x,y) \end{aligned}$$

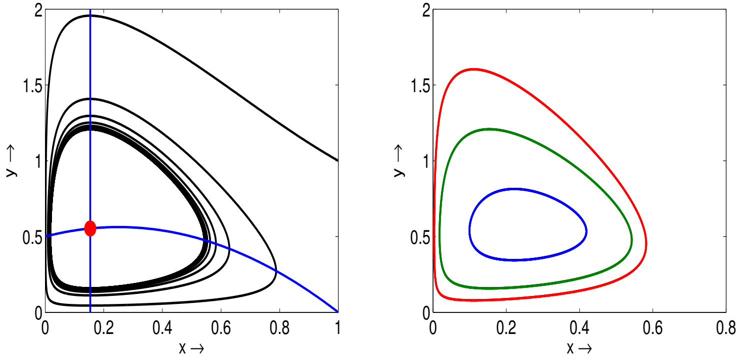
Preliminary results

(i) Positivity and boundedness of solutions;
(ii) Determination of equilibria and their stability;
(iii) Stability and LOCAL bifurcation analysis;
(iv) Global stability;

Basic prey-predator models
Rosenzweig-MacArthur model



Basic prey-predator models
Attractors and the risk of extinction in RM model



Prey population can go to extinct which in turn drives the predators to extinction.

Basic prey-predator models
Prey and their predators

Vole, Snowshoe Hare, Lemmings and their predators

Basic prey-predator models
Gause type Models

Gause type model

$$\frac{du}{dt} = uf(u) - p(u)v, \quad \frac{dv}{dt} = ep(u)v - q(v)v$$
$$\frac{du}{dt} = uf(u) - p(u,v)v, \quad \frac{dv}{dt} = ep(u,v)v - q(v)v$$

Variables, parameter and functions

- u, v $\rightarrow$ prey, predator population
- t $\rightarrow$ time
- $f(u)$ $\rightarrow$ per capita prey growth rate
- $p(u)$ $\rightarrow$ prey-dependent functional response
- $p(u,v)$ $\rightarrow$ prey and predator-dependent functional response
- $q(v)$ $\rightarrow$ per capita predator death rate
- e $\rightarrow$ conversion efficiency

Basic prey-predator models

Growth functions

- $f(u) = (1-u) \rightarrow$ Logistic growth

Functional responses

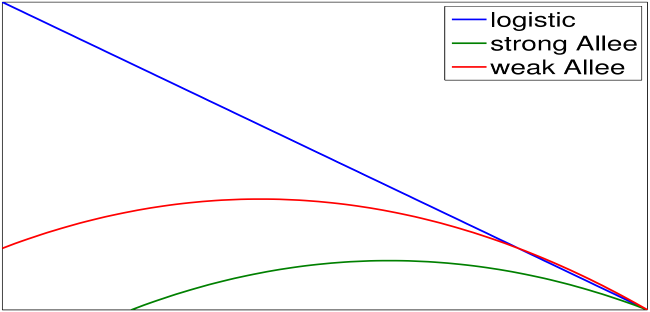
- $p(u) = \dfrac{u}{\beta + u} \rightarrow$ Holling type II
- $p(u) = \dfrac{u^2}{\xi + u^2} \rightarrow$ Holling type III
- $p(u) = \dfrac{u}{\eta + u^2} \rightarrow$ Monode-Haldane
- $p(u, v) = \dfrac{u}{u + v} \rightarrow$ ratio-dependent
- $p(u, v) = \dfrac{u}{\beta + u + v} \rightarrow$ Beddington-DeAngelis

Basic prey-predator models
Species showing Allee effect




Bluefin Tuna, Kakapo bird, fruit flies, African wild dogs

Basic prey-predator models
Per capita prey growth functions



$f(u) = (1-u)(u-\eta)$,
$0 < \eta < 1 \rightarrow$ strong Allee effect
$-1 < \eta < 0 \rightarrow$ weak Allee effect

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Delayed models
Existing Models

Math. Model. Nat. Phenom.
Vol. 4, No. 2, 2009, pp. 140-188
DOI: 10.1051/mmnp/20094207

On Nonlinear Dynamics of Predator-Prey Models with Discrete Delay*

S. Ruan[†]

Department of Mathematics, University of Miami, Coral Gables, FL 33124-4250, USA

Delayed model with Hutchinson type delay

$$\begin{aligned}\frac{du(t)}{dt} &= u(t)(1-u(t-\tau)) - \frac{u(t)v(t)}{\beta+u(t)},\\ \frac{dv(t)}{dt} &= \frac{\alpha u(t)v(t)}{\beta+u(t)} - \delta v(t)\end{aligned}$$

Delayed models
Existing Models

Bulletin of Mathematical Biology Vol. 45, No. 6, pp. 991–1004, 1983.
Printed in Great Britain

0092–8240/83$3.00 + 0.00
Pergamon Press Ltd.

THE TRADE-OFF BETWEEN MUTUAL INTERFERENCE AND TIME LAGS IN PREDATOR–PREY SYSTEMS†

H. I. Freedman and V. Sree Hari Rao‡
Department of Mathematics,
University of Alberta,
Edmonton, Alberta, Canada T6G 2G1

Model with Gestation Delay

$$\begin{aligned}
\frac{du(t)}{dt} &= u(t)(1-u(t)) - \frac{u(t)v(t)}{\beta+u(t)}, \\
\frac{dv(t)}{dt} &= \frac{\alpha u(t-\tau)v(t)}{\beta+u(t-\tau)} - \delta v(t)
\end{aligned}$$

Delayed models
Existing Models

TIME LAG IN PREY-PREDATOR POPULATION MODELS

PETER J. WANGERSKY AND W. J. CUNNINGHAM
Osborn Zoological Laboratory and Dunham Laboratory of Electrical Engineering, Yale University, New Haven, Connecticut

Ecology, **38**, 136 - 139 (1957)

Delayed model with Wangersky-Cunningham formulation

$$\begin{aligned} \frac{du(t)}{dt} &= u(t)(1-u(t)) - \frac{u(t)v(t)}{\beta+u(t)}, \\ \frac{dv(t)}{dt} &= \frac{\alpha u(t-\tau)v(t-\tau)}{\beta+u(t-\tau)} - \delta v(t) \end{aligned}$$

Delayed models
Existing Models

J. Math. Biol. 49, 188–200 (2004)
Digital Object Identifier (DOI):
10.1007/s00285-004-0278-2

Mathematical Biology

Stephen A. Gourley · Yang Kuang

A stage structured predator-prey model and its dependence on maturation delay and death rate

Gourley-Kuang formulation

$$\begin{aligned}\frac{du(t)}{dt} &= u(t)f(u(t)) - p(u(t))v(t),\\ \frac{dv(t)}{dt} &= e^{-\delta_j\tau}p(u(t-\tau))v(t-\tau) - \delta v(t)\end{aligned}$$

Delayed models
Model with maturation delay

Reported results

- Dealy induced destabilization
- Destabilization through Hopf-bifurcation
- Destabilization leading to complex/chaotic dynamics
- Stability of Hopf-bifurcating periodic solution
- Switching of stability

Interesting questions

Why, when and how destabilization takes place due to discrete time delay?
Does discrete delay always destabilize the system?

Delayed models
Targeted models

Model with logistic growth

$$\begin{aligned}\frac{du(t)}{dt} &= u(t)(1-u(t)) - p(u(t),v(t))v(t),\\ \frac{dv(t)}{dt} &= e^{-\delta_j\tau}p(u(t-\tau),v(t-\tau))v(t-\tau) - \delta v(t)\end{aligned}$$

Model with Allee effect in prey growth

$$\begin{aligned}\frac{du(t)}{dt} &= u(t)(1-u(t))(u(t)-\beta) - p(u(t),v(t))v(t),\\ \frac{dv(t)}{dt} &= e^{-\delta_j\tau}p(u(t-\tau),v(t-\tau))v(t-\tau) - \delta v(t)\end{aligned}$$

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RM model with strong Allee effect

RM model with multiplicative Allee effect

$$\begin{aligned}\frac{dx}{dt} &= x(1-x)(x-\beta) - \frac{xy}{x+\alpha}\\ \frac{dy}{dt} &= \frac{\gamma xy}{x+\alpha} - \delta y\end{aligned}$$

$0 < \beta < 1 \rightarrow$ strong Allee effect
$-1 < \beta < 0 \rightarrow$ weak Allee effect

Axial Equilibria

$E_0(0,0) \rightarrow$ always stable
$E_1(\beta,0) \rightarrow$ saddle point if $\gamma < \delta(\alpha+\beta)/\beta$ and is unstable node otherwise
$E_2(1,0) \rightarrow$ LAS for $\gamma < \delta(1+\alpha)$, saddle point otherwise

RM model with strong Allee effect

Coexisting equilibrium E_*

$$x_* = \frac{\alpha\delta}{\gamma - \delta}, \quad y_* = (1 - x_*)(x_* - \beta)(x_* + \alpha)$$

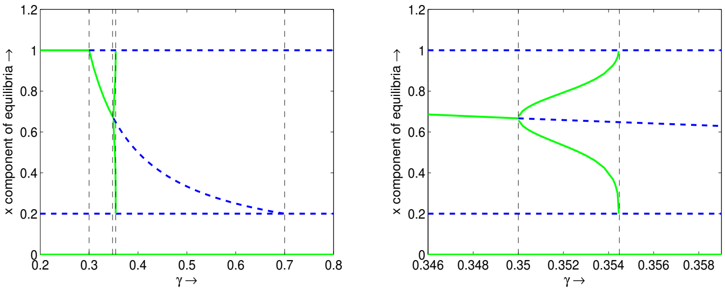
Stability and bifurcation of E_*

E_* appears and disappears through transcritical bifurcations, TCB thresholds are $\gamma_{TC_1} \equiv \delta(1+\alpha)$ and $\gamma_{TC_2} \equiv \delta(\alpha+\beta)/\beta$ respectively.

E_* is LAS for $\gamma_{TC_1} < \gamma < \gamma_H$ where

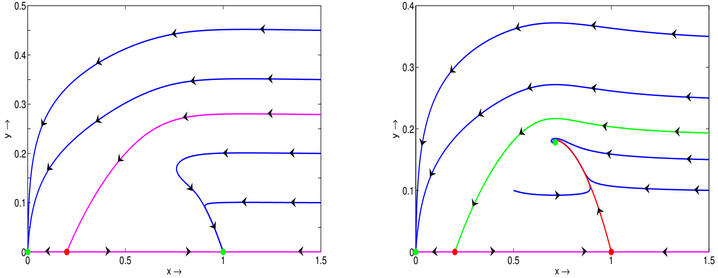
$$\gamma_H = \delta + \frac{3\alpha\delta}{\beta - \alpha + 1 + \sqrt{(\alpha+\beta)^2 + (\alpha+1)(1-\beta)}}.$$

RM model with strong Allee effect



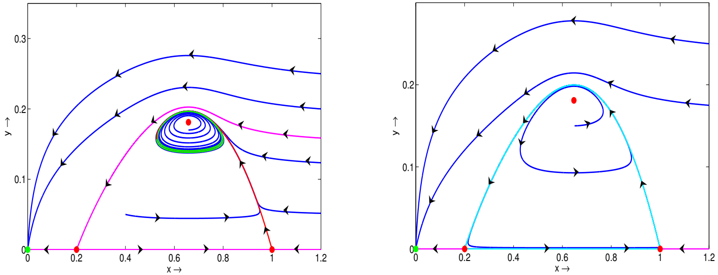
Bifurcation diagram for $\beta = 0.2$, $\alpha = 0.5$, $\delta = 0.2$ and with γ as bifurcation parameter.

RM model with strong Allee effect



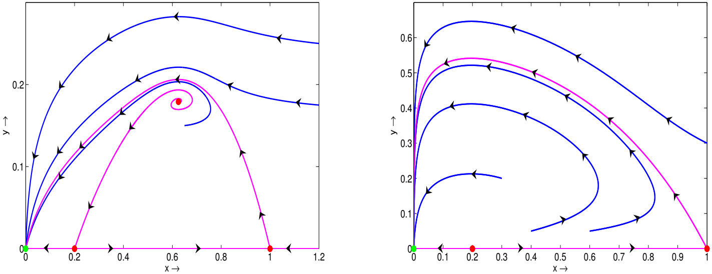
(Left) $\gamma = 0.29 < \gamma_{TC1}$, **(Right)** $\gamma_{TC1} < \gamma = 0.34 < \gamma_H$

RM model with strong Allee effect



(Left) $\gamma_H < \gamma = 0.352 < \gamma_{HC}$, **(Right)** $\gamma = 0.3544596588 = \gamma_{HC}$

RM model with strong Allee effect



(Left) $\gamma_{HC} < \gamma = 0.36 < \gamma_{TC2}$, **(Right)** $\gamma_{TC_2} < \gamma = 0.71$

Models with Allee effects

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Available online at www.sciencedirect.com

ScienceDirect

Mathematical Biosciences 209 (2007) 451–469

Mathematical Biosciences

www.elsevier.com/locate/mbs

Heteroclinic orbits indicate overexploitation in predator–prey systems with a strong Allee effect

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[a] Department of Theoretical Biology, Faculty of Earth and Life Sciences, Vrije Universiteit, de Boelelaan 1087, 1081 HV Amsterdam, The Netherlands

[b] Wageningen University and Research Centre, Biometris, P.O. Box 100, 6700 AC Wageningen, The Netherlands

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Ecological Complexity

journal homepage: www.elsevier.com/locate/ecocom

Original research article

Bifurcation analysis of a ratio-dependent prey–predator model with the Allee effect

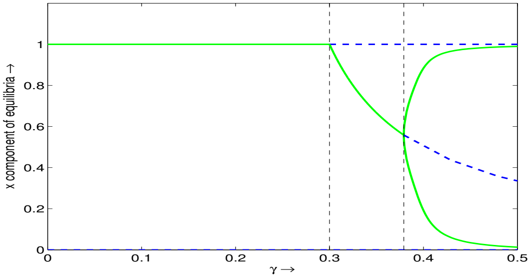
Moitri Sen [a], Malay Banerjee [a], Andrew Morozov [b,c,*]

[a] Department of Mathematics and Statistics, I. I. T. Kanpur, India
[b] Department of Mathematics, University of Leicester, UK
[c] Shirshov Institute of Oceanology, Moscow, Russia

RM model with weak Allee effect

Model with weak Allee effect ($-1 < \beta < 0$)

$$\frac{dx}{dt} = x(1-x)(x-\beta) - \frac{xy}{x+\alpha}$$
$$\frac{dy}{dt} = \frac{\gamma xy}{x+\alpha} - \delta y$$



Outline

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4 Effect of maturation delay

Effect of maturation delay
Delayed RM model with Allee effect

Journal of Theoretical Biology 412 (2017) 154–171

Contents lists available at ScienceDirect

Journal of Theoretical Biology

journal homepage: www.elsevier.com/locate/yjtbi

Maturation delay for the predators can enhance stable coexistence for a class of prey–predator models

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[a] Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Kanpur 208016, India
[b] College of Science and Engineering, Aoyama Gakuin University, Kanagawa, Japan

Basic Model

$$
\begin{aligned}
\frac{dx(t)}{dt} &= x(t)(1-x(t))(x(t)-\beta) - \frac{x(t)y(t)}{x(t)+\alpha},\\
\frac{dy(t)}{dt} &= \frac{\gamma e^{-\delta_j \tau} x(t-\tau)y(t-\tau)}{x(t-\tau)+\alpha} - \delta y(t).
\end{aligned}
$$

Effect of maturation delay
Delayed RM model with strong Allee effect

Model with Maturation Delay ($0 < \beta < 1$)

$$
\begin{aligned}
\frac{dx(t)}{dt} &= x(t)(1-x(t))(x(t)-\beta) - \frac{x(t)y_a(t)}{x(t)+\alpha}, \\
\frac{dy_j(t)}{dt} &= \frac{\gamma x(t)y_a(t)}{x(t)+\alpha} - \frac{\gamma e^{-\delta_j \tau} x(t-\tau)y_a(t-\tau)}{x(t-\tau)+\alpha} - \delta_j y_j(t), \\
\frac{dy_a(t)}{dt} &= \frac{\gamma e^{-\delta_j \tau} x(t-\tau)y_a(t-\tau)}{x(t-\tau)+\alpha} - \delta y_a(t),
\end{aligned}
$$

Effect of maturation delay
Delayed RM model with strong Allee effect

Model with Maturation Delay ($0 < \beta < 1$)

$$\begin{aligned}
\frac{dx(t)}{dt} &= x(t)(1-x(t))(x(t)-\beta) - \frac{x(t)y_a(t)}{x(t)+\alpha}, \\
\frac{dy_j(t)}{dt} &= \frac{\gamma x(t)y_a(t)}{x(t)+\alpha} - \frac{\gamma e^{-\delta_j\tau}x(t-\tau)y_a(t-\tau)}{x(t-\tau)+\alpha} - \delta_j y_j(t), \\
\frac{dy_a(t)}{dt} &= \frac{\gamma e^{-\delta_j\tau}x(t-\tau)y_a(t-\tau)}{x(t-\tau)+\alpha} - \delta y_a(t),
\end{aligned}$$

Justification for $e^{-\delta_j\tau}$

$$\frac{dy_j(t)}{dt} = -\delta_j y_j$$

$$y_j(\tau) = e^{-\delta_j\tau}y_j(0)$$

Effect of maturation delay
Delayed RM model with strong Allee effect

Simplication of the model (2nd equation)

$$\frac{d}{dt}\left(e^{\delta_j t} y_j(t)\right) = e^{\delta_j t}\frac{\gamma x(t) y_a(t)}{\alpha + u(t)} - e^{\delta_j (t-\tau)}\frac{\gamma x(t-\tau) y_a(t-\tau)}{\alpha + x(t-\tau)}$$

$$y_j(t) = \gamma \int_{-\tau}^{0} e^{\delta_j s}\frac{x(s+t) y_a(s+t)}{\alpha + x(s+t)} ds$$

Effect of maturation delay
Delayed RM model with strong Allee effect

Simplication of the model (2nd equation)

$$\frac{d}{dt}\left(e^{\delta_j t} y_j(t)\right) = e^{\delta_j t}\frac{\gamma x(t) y_a(t)}{\alpha + u(t)} - e^{\delta_j (t-\tau)}\frac{\gamma x(t-\tau) y_a(t-\tau)}{\alpha + x(t-\tau)}$$

$$y_j(t) = \gamma \int_{-\tau}^{0} e^{\delta_j s}\frac{x(s+t) y_a(s+t)}{\alpha + x(s+t)} ds$$

Final model

$$\frac{dx(t)}{dt} = x(t)(1 - x(t))(x(t) - \beta) - \frac{x(t)y(t)}{x(t)+\alpha},$$

$$\frac{dy(t)}{dt} = \frac{\gamma e^{-\delta_j \tau} x(t-\tau) y(t-\tau)}{x(t-\tau)+\alpha} - \delta y(t).$$

Effect of maturation delay
Delayed RM model with strong Allee effect

Simplication of the model (2nd equation)

$$\frac{d}{dt}\left(e^{\delta_j t} y_j(t)\right) = e^{\delta_j t}\frac{\gamma x(t) y_a(t)}{\alpha + u(t)} - e^{\delta_j (t-\tau)}\frac{\gamma x(t-\tau) y_a(t-\tau)}{\alpha + x(t-\tau)}$$

$$y_j(t) = \gamma \int_{-\tau}^{0} e^{\delta_j s}\frac{x(s+t) y_a(s+t)}{\alpha + x(s+t)} ds$$

Final model

$$\frac{dx(t)}{dt} = x(t)(1 - x(t))(x(t) - \beta) - \frac{x(t)y(t)}{x(t)+\alpha},$$

$$\frac{dy(t)}{dt} = \frac{\gamma e^{-\delta_j \tau} x(t-\tau) y(t-\tau)}{x(t-\tau)+\alpha} - \delta y(t).$$

$$y_j(0) = \gamma \int_{-\tau}^{0} e^{\delta_j s} y_a(s) p(x(s)) ds.$$

Effect of maturation delay
Delayed RM model with strong Allee effect

Coexisting equilibrium ($0 < \beta < 1$)

$$x_*(\tau) \equiv \frac{\alpha\delta}{\gamma e^{-\delta_j \tau} - \delta}, \quad y_*(\tau) = (1 - x_*(\tau))(x_*(\tau) - \beta)(x_*(\tau) + \alpha).$$

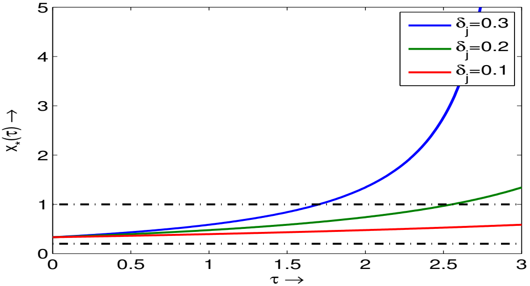
$$\frac{\delta(\alpha + 1)}{\gamma} < e^{-\delta_j \tau} < \frac{\delta(\alpha + \beta)}{\beta\gamma}.$$

Effect of maturation delay
Delayed RM model with strong Allee effect

Coexisting equilibrium ($0 < \beta < 1$)

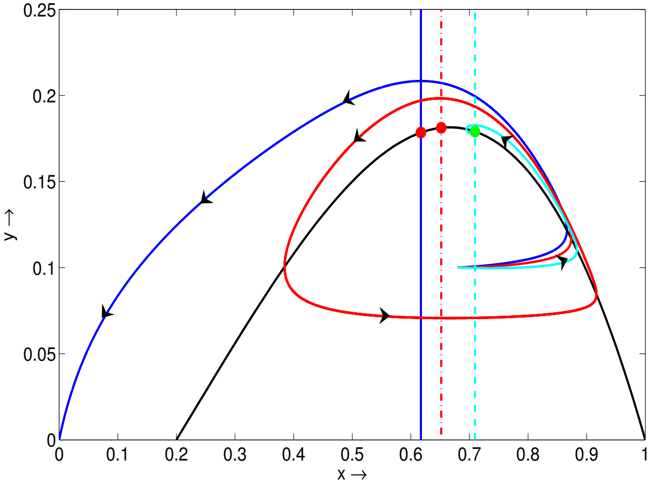
$$x_*(\tau) \equiv \frac{\alpha\delta}{\gamma e^{-\delta_j \tau} - \delta}, \quad y_*(\tau) = (1 - x_*(\tau))(x_*(\tau) - \beta)(x_*(\tau) + \alpha).$$

$$\frac{\delta(\alpha+1)}{\gamma} < e^{-\delta_j \tau} < \frac{\delta(\alpha+\beta)}{\beta\gamma}.$$



$\alpha = 0.5$, $\beta = 0.2$, $\delta = 0.2$, $\gamma = 0.5$

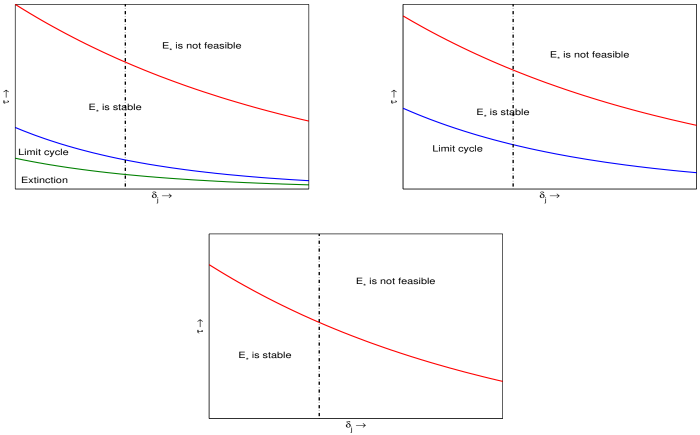
Effect of maturation delay
Delayed RM model with strong Allee effect



$\alpha = 0.5,\ \beta = 0.2,\ \gamma = 0.362,\ \delta = 0.2,\ \delta_j = 0.1$

$\tau = 0$ (blue), $\tau = 0.24$ (red) and $\tau = 0.6$ (cyan)

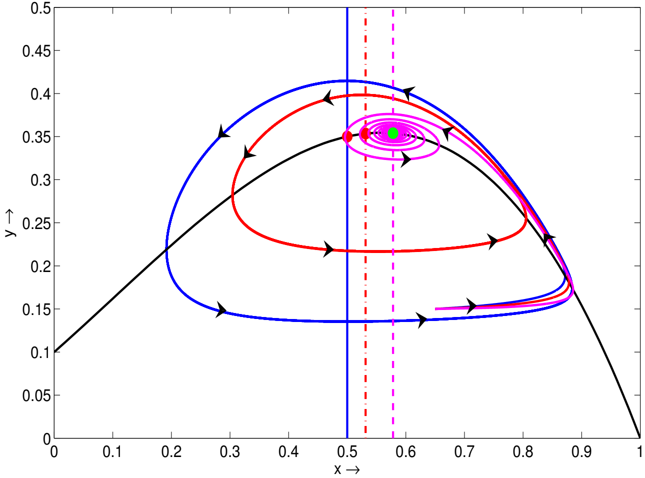
Effect of maturation delay
Delayed RM model with strong Allee effect



$$\gamma_{HC} < \gamma < \gamma_{TC_2};\ \gamma_H < \gamma < \gamma_{HC};\ \gamma_{TC_1} < \gamma < \gamma_H$$

$\delta = \delta_j \rightarrow$ vertical dotted line

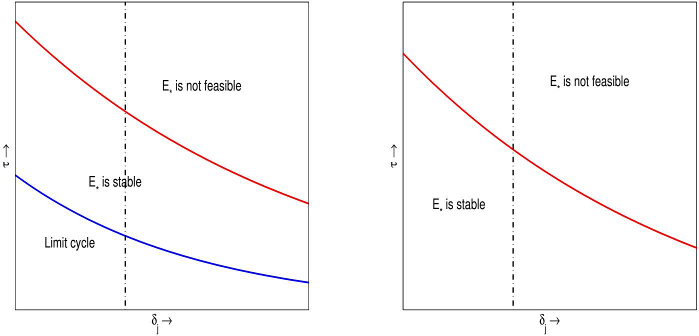
Effect of maturation delay
Delayed RM model with weak Allee effect



$\alpha = 0.5$, $\beta = -0.2$, $\gamma = 0.4$, $\delta = 0.2$, $\delta_j = 0.1$

$\tau = 0$ (blue), $\tau = 0.3$ (red) and $\tau = 0.8$ (magenta)

Effect of maturation delay
Delayed RM model with weak Allee effect



$$\gamma_H < \gamma < \gamma_{HC};\ \gamma_{TC_1} < \gamma < \gamma_H$$
$\delta = \delta_j \rightarrow$ vertical dotted line

Effect of maturation delay
Model with ecological data- Lemming-Stoat interaction

Mathematical Biosciences 221 (2009) 1–10

Contents lists available at ScienceDirect

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Mathematical Biosciences

journal homepage: www.elsevier.com/locate/mbs

The roles of predator maturation delay and functional response in determining the periodicity of predator–prey cycles

Hao Wang [a,*], John D. Nagy [b,*], Olivier Gilg [c], Yang Kuang [d]

[a] Department of Mathematical and Statistical Sciences, University of Alberta, Canada
[b] Department of Life Sciences, Scottsdale Community College, School of Life Sciences, Arizona State University, USA
[c] Department of Biological and Environmental Sciences, University of Helsinki, Finland
[d] Department of Mathematics and Statistics, Arizona State University, USA

Effect of maturation delay
Ratio-dependent model with Allee effect

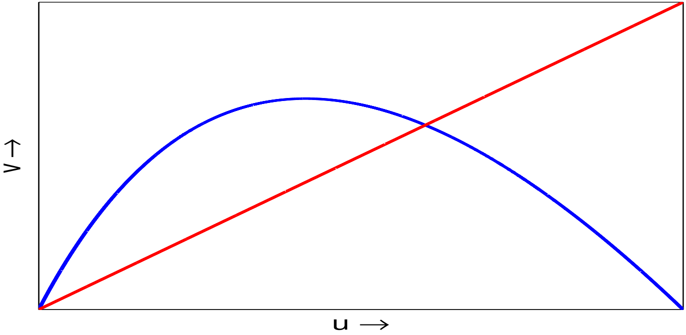
Ratio-dependent functional response

$$\frac{du}{dt} = u(1-u)(u-\beta) - \frac{\alpha uv}{u+v}, \quad \frac{dv}{dt} = \frac{\gamma uv}{u+v} - \delta v$$

Effect of maturation delay
Ratio-dependent model with Allee effect

Ratio-dependent functional response

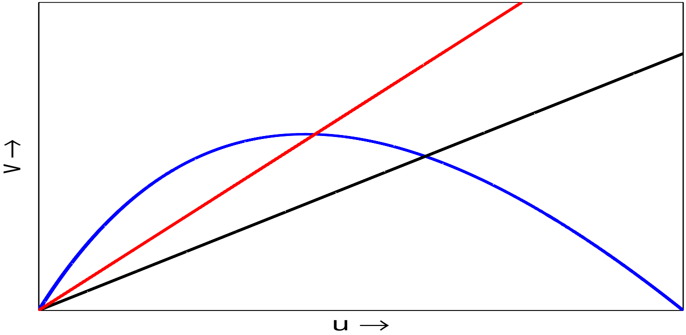
$$\frac{du}{dt} = u(1-u)(u-\beta) - \frac{\alpha uv}{u+v}, \quad \frac{dv}{dt} = \frac{\gamma uv}{u+v} - \delta v$$



Effect of maturation delay
Ratio-dependent model with Allee effect

Ratio-dependent functional response

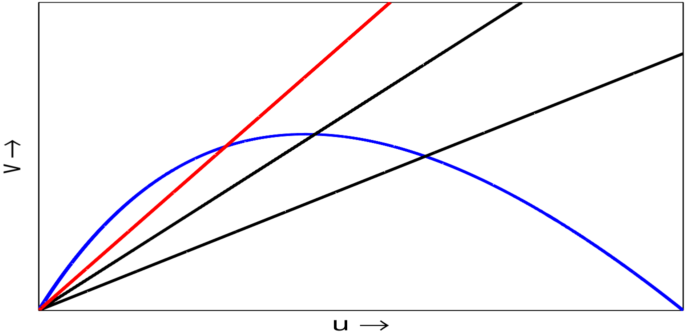
$$\frac{du}{dt} = u(1-u)(u-\beta) - \frac{\alpha uv}{u+v}, \quad \frac{dv}{dt} = \frac{\gamma uv}{u+v} - \delta v$$



Effect of maturation delay
Ratio-dependent model with Allee effect

Ratio-dependent functional response

$$\frac{du}{dt} = u(1-u)(u-\beta) - \frac{\alpha uv}{u+v}, \quad \frac{dv}{dt} = \frac{\gamma uv}{u+v} - \delta v$$



Effect of maturation delay
Delayed ratio-dependent model with Allee effect

Model

$$\begin{aligned}\frac{du(t)}{dt} &= u(t)(1-u(t))(u(t)-\beta) - \frac{\alpha u(t)v(t)}{u(t)+v(t)},\\ \frac{dv(t)}{dt} &= \frac{\gamma e^{-\delta_j\tau} u(t-\tau)v(t-\tau)}{u(t-\tau)+v(t-\tau)} - \delta v(t).\end{aligned}$$

Effect of maturation delay
Delayed ratio-dependent model with Allee effect

Model

$$\frac{du(t)}{dt} = u(t)(1-u(t))(u(t)-\beta) - \frac{\alpha u(t)v(t)}{u(t)+v(t)},$$

$$\frac{dv(t)}{dt} = \frac{\gamma e^{-\delta_j \tau} u(t-\tau)v(t-\tau)}{u(t-\tau)+v(t-\tau)} - \delta v(t).$$

Ecological Complexity 19 (2014) 23–34

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Ecological Complexity

journal homepage: www.elsevier.com/locate/ecocom

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Original Research Article

Stage-structured ratio-dependent predator–prey models revisited: When should the maturation lag result in systems' destabilization?

Moitri Sen [a], Malay Banerjee [a], Andrew Morozov [b,*]

[a] Department of Mathematics and Statistics, I.I.T. Kanpur, Kanpur 208016, India
[b] Department of Mathematics, University of Leicester, UK

Effect of maturation delay
Model with Beddington-DeAngelis functional response

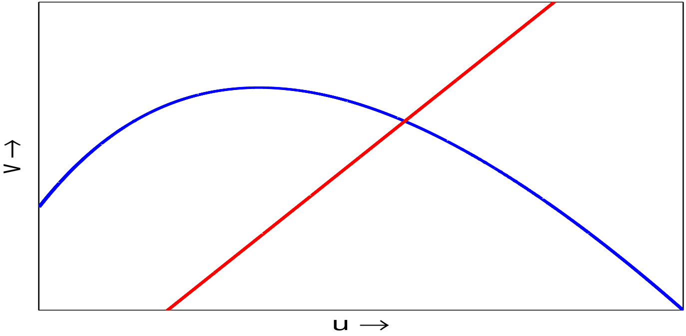
Beddington-DeAngelis functional response

$$\frac{du}{dt} = u(1-u) - \frac{\alpha uv}{\beta + u + v}, \quad \frac{dv}{dt} = \frac{\gamma uv}{\beta + u + v} - \delta v$$

Effect of maturation delay
Model with Beddington-DeAngelis functional response

Beddington-DeAngelis functional response

$$\frac{du}{dt} = u(1-u) - \frac{\alpha uv}{\beta + u + v}, \quad \frac{dv}{dt} = \frac{\gamma uv}{\beta + u + v} - \delta v$$



Effect of maturation delay
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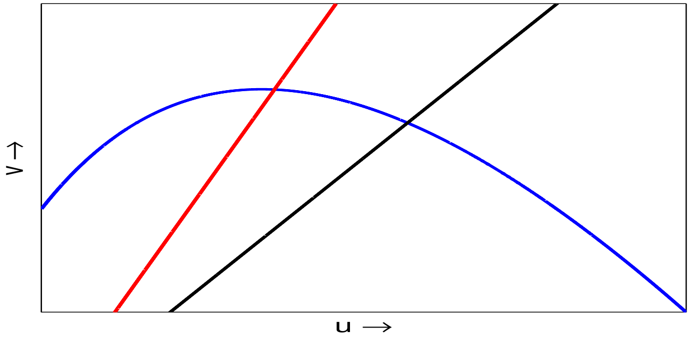

Effect of maturation delay
Model with Beddington-DeAngelis functional response

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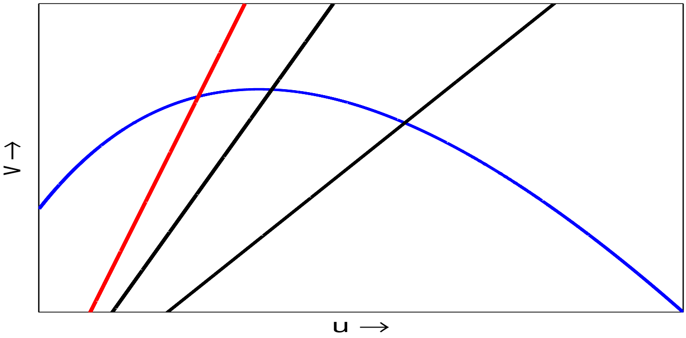

Effect of maturation delay
Model with Beddington-DeAngelis functional response

Beddington-DeAngelis functional response

$$\frac{du(t)}{dt} = u(t)(1-u(t)) - \frac{\alpha u(t)v(t)}{\beta + u(t) + v(t)},$$
$$\frac{dv(t)}{dt} = \frac{\gamma e^{-\delta_j \tau} u(t-\tau)v(t-\tau)}{\beta + u(t-\tau) + v(t-\tau)} - \delta v(t)$$

Effect of maturation delay
Model with Beddington-DeAngelis functional response

Beddington-DeAngelis functional response

$$\frac{du(t)}{dt} = u(t)(1-u(t)) - \frac{\alpha u(t)v(t)}{\beta + u(t) + v(t)},$$

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Predator–prey model of Beddington–DeAngelis type with maturation and gestation delays

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Thank You