New prospects for the numerical bifurcation analysis of nonlinear delay equations

Francesca Scarabel*§, Odo Diekmann†, Mats Gyllenberg§, Rossana Vermiglio‡

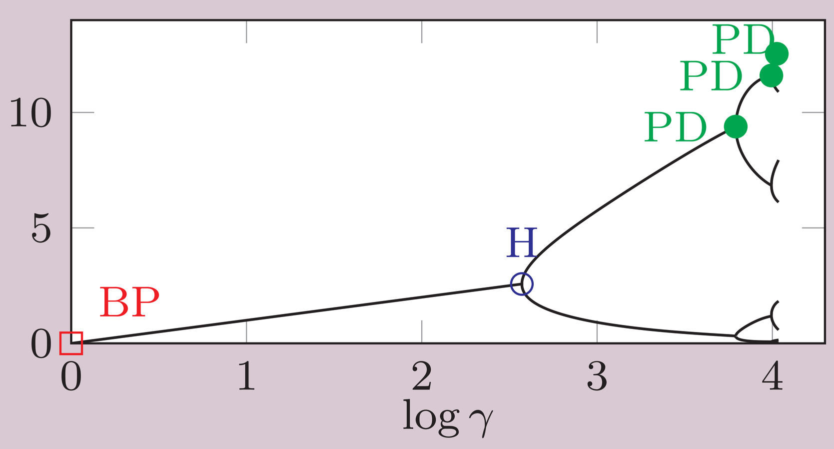
§University of Helsinki (Finland), †Utrecht University (the Netherlands), ‡University of Udine (Italy),

*francesca.scarabel@helsinki.fi

Renewal equation

a simplified model for cannibalism, [3]

$$x(t) = \frac{\gamma}{2}\int_1^3 x(t-a)e^{-x(t-a)}\,\mathrm{d}a$$



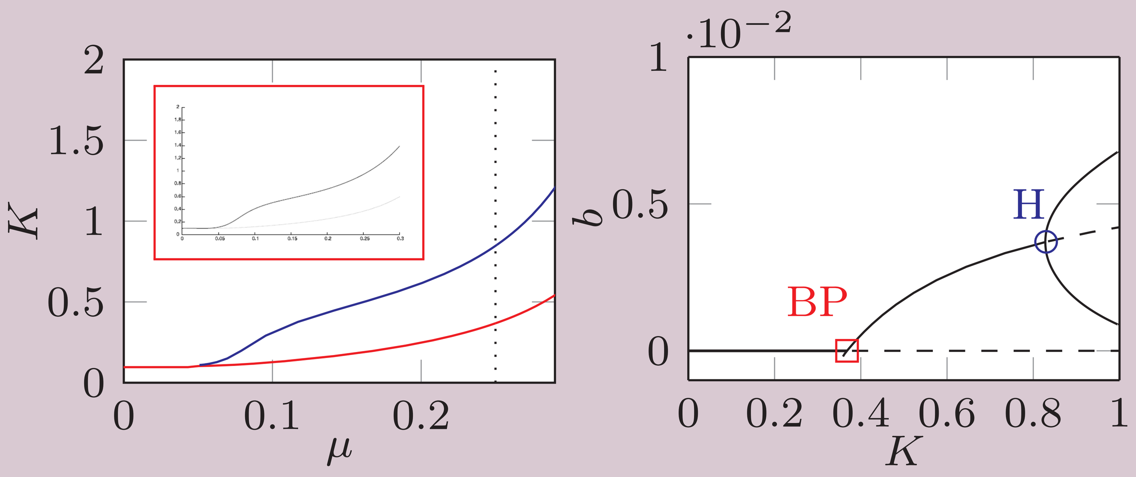
Bifurcation diagram w.r.t. γ, with detection of a period doubling cascade, $M = 11$.

Daphnia-type model

a physiologically structured model with individual growth and maximal age, [5]

$$\begin{cases} b(t) = \int_{\bar{a}(S_t)}^{h} \beta(X(a,S_t),S_t)e^{-\mu a}b(t-a)\,\mathrm{d}a \\ \dot{S}(t) = f(S(t)) - \int_0^h \gamma(X(a,S_t),S_t)e^{-\mu a}b(t-a)\,\mathrm{d}a \end{cases}$$

$$x'(\alpha) = g(x(\alpha), S_t(\alpha - a))$$
$$x(0) = x_b, \;\; X(a, S_t) = x(a)$$



Left: Existence (red) and stability (blue) boundary of the nontrivial equilibrium in (μ, K); the red-boxed reference diagram is taken from [5].
Right: Bifurcation diagram of b w.r.t. K, $\mu = 0.25$, $M = 7$.

INPUT:
nonlinear delay equation

APPROXIMATION WITH ODEs
+
SOFTWARE FOR ODEs

OUTPUT:
bifurcation analysis

Delay differential equation

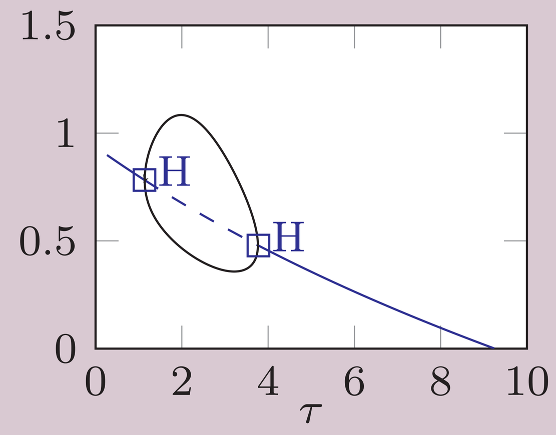
an age structured model with continuous reproduction and no maximal age, [1]

$$\dot{N}(t) = -dN(t) + d\int_0^\infty F^{(n)}_{n/\tau}(s)e^{\eta - d_J s - aN(t-s)}N(t-s)\,\mathrm{d}s$$

where $F^{(n)}_\alpha(s)$ is the Gamma distribution

$$F^{(n)}_\alpha(s) = \frac{\alpha^n s^{n-1}e^{-\alpha s}}{(n-1)!}, \qquad s \geq 0, \alpha > 0, n \in \mathbb{N}$$

$d = 0.5$, $d_J = 1$, $a = 7$, $b = 350$, $n = 10$.
Bifurcation diagram w.r.t. τ, $M = 20$.



Daphnia-type model

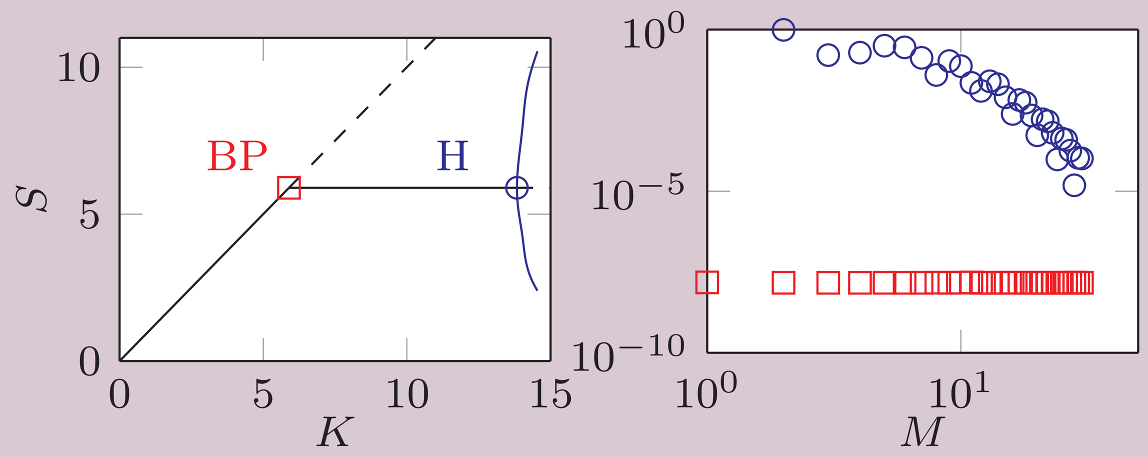
a physiologically structured model with no maximal age, [4]

$$\begin{cases} b(t) = \frac{\alpha S(t)}{1+S(t)}N(t;S_t) \\ \dot{S}(t) = rS(t)\left(1 - \frac{S(t)}{K}\right) - \frac{S(t)}{1+S(t)}N(t;S_t) \end{cases}$$

with

$$N(t;S_t) = \int_0^\infty e^{-\mu a}b(t-a)\left(\int_{t-a}^t \frac{S(\sigma)}{1+S(\sigma)}e^{\sigma - t}\,\mathrm{d}\sigma\right)^2 \mathrm{d}a$$

$\mu = 0.5$, $\alpha = 1.5$, $r = 3$
Left: bifurcation diagram of S w.r.t. K, $M = 10$.
Right: error in BP and H, Matcont tolerance 10^{-6}.



Pseudospectral method [2]

The problem

Let $\tau > 0$ be the maximum delay. Then

if $\tau < \infty$, let $\rho = 0$ and $\mathcal{I} = [-\tau, 0]$
if $\tau = \infty$, let $\rho > 0$ and $\mathcal{I} = (-\infty, 0]$

Denote $x_t(\theta) := x(t+\theta)$, $\theta \in \mathcal{I}$ and define

$$X = L^1_\rho(\mathcal{I},\mathbb{R}) := \{\varphi\colon \mathcal{I} \to \mathbb{R} \mid \int_{\mathcal{I}} e^{\rho\theta}|\varphi(\theta)|d\theta < \infty\}$$

$$Y = C_{0,\rho}(\mathcal{I},\mathbb{R}) := \{\psi\colon \mathcal{I} \to \mathbb{R} \mid \lim_{\theta\to-\infty} e^{\rho\theta}\psi(\theta) = 0 \text{ and } \sup_{\theta\in\mathcal{I}} e^{\rho\theta}|\psi(\theta)| < \infty\}$$

A generic nonlinear system of coupled equations is

$$\begin{cases} x(t) = F(x_t, y_t), & x_0 = \varphi \\ \dot{y}(t) = G(x_t, y_t), & y_0 = \psi \end{cases}$$

for $t \geq 0$, with $x_t \in X$, $y_t \in Y$, and

$$F, G\colon X \times Y \to \mathbb{R}.$$

Motivated by results from approximation theory, if $\tau = \infty$ we approximate the mapped states

$$e^{\rho\theta}x_t(\theta), \quad e^{\rho\theta}y_t(\theta), \qquad \theta \in \mathcal{I}.$$

Pseudospectral discretization

Mesh of $M+1$ points in $\mathcal{I}$,

$$-\tau \leq \theta_M < \theta_{M-1} < \cdots < \theta_0 = 0$$

Lagrange basis and differentiation matrix,

$$\ell_j(\theta) = \prod_{k\neq j}\frac{\theta - \theta_k}{\theta_j - \theta_k}, \quad j = 0, \ldots, M$$

$$(\hat{d}_M)_j = e^{\rho\theta_j}\ell_0'(\theta_j), \; (\hat{D}_M)_{jk} = \frac{e^{\rho\theta_j}}{e^{\rho\theta_k}}\ell_k'(\theta_j), \quad j,k > 0$$

Weighted interpolation of $\mathrm{x} \in \mathbb{R}$, $\Phi \in \mathbb{R}^M$,

$$\hat{p}_M(\mathrm{x},\Phi)(\theta) = e^{\rho\theta}\ell_0(\theta)\,\mathrm{x} + \sum_{k=1}^M e^{\rho\theta}\ell_k(\theta)\frac{\Phi_k}{e^{\rho\theta_k}}, \quad \theta \in \mathcal{I}$$

Let $x_M, y_M \in \mathbb{R}$ and $U_M, V_M \in \mathbb{R}^M$ s.t., for $j > 0$

$$x_M(t) \approx x(t), \quad U_{M,j}(t) \approx e^{\rho\theta_j}x(t+\theta_{M,j}),$$
$$y_M(t) \approx y(t), \quad V_{M,j}(t) \approx e^{\rho\theta_j}y(t+\theta_{M,j})$$

The approximating ODE system in $\mathbb{R}^{2M+1}$ is

$$\begin{cases} U_M' = \hat{d}_M x_M + \hat{D}_M U_M - \rho U_M \\ y_M' = G(e^{-\rho\cdot}\hat{p}_M(x_M,U_M), e^{-\rho\cdot}\hat{p}_M(y_M,V_M)) \\ V_M' = \hat{d}_M y_M + \hat{D}_M V_M - \rho V_M \end{cases}$$

where x_M is implicitly defined by

$$x_M = F(e^{-\rho\cdot}\hat{p}_M(x_M,U_M), e^{-\rho\cdot}\hat{p}_M(y_M,V_M))$$

Results

In the examples above, the approximating ODE systems were analyzed with the continuation package Matcont for Matlab, for a certain discretization index M. The figures show the results of the numerical bifurcation analysis when varying some parameters of the systems.

Concluding remarks

Applicable to:

- Volterra integral equations, integro-differential equations, coupled equations
- nonlinear equations: no need to linearize
- discrete, distributed and unbounded delays
- state-dependent discontinuities (e.g. maturation)

Other advantages (see [2, 3]):

- easy ODE formulation
- evidence of spectral convergence, $O(M^{-M})$
- one-to-one correspondence of equilibria
- exploits available software for ODEs

Problematic issues:

- computational time due to external ODEs and other complexities of the Daphnia model
- approximation of non-smooth solutions

References

[1] E. Beretta and D. Breda (2016), Discrete or distributed delay? Effects on stability of population growth, *Math. Biosci. Eng.*, 13(1):19–41.

[2] D. Breda, O. Diekmann, M. Gyllenberg, F. Scarabel and R. Vermiglio (2016), Pseudospectral discretization of nonlinear delay equations: new prospects for numerical bifurcation analysis, *SIAM J. Appl. Dyn. Syst.*, 15(1): 1–23.

[3] D. Breda, O. Diekmann, D. Liessi and F. Scarabel (2016), Numerical bifurcation analysis of a class of nonlinear renewal equations, *Electron. J. Qual. Theory Differ. Equ.*, 65: 1–24.

[4] A.M. de Roos (1989), Daphnids on a train: Development and application of a new numerical method for physiologically structured population models, PhD thesis, Leiden University.

[5] A.M. de Roos, O. Diekmann, P. Getto and M.A. Kirkilionis (2010), Numerical equilibrium analysis for structured consumer resource models, *Bull. Math. Biol.*, 72: 259–297.